Institution of Electrical Engineers.

ORIGINAL COMMUNICATION.

DIMENSIONS OF SEMI-PHASE CORE AND SHELL TRANSFORMERS, WITH COPPER AND IRON CIRCUITS OF UNIFORM RECTANGULAR CROSS-SECTION.

BY

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SECTION I.

Symbols used

$P_o = \frac{1}{2}$ (input and output).
$p_i =$ Iron loss as percentage of P_o.
$p_c =$ Copper loss as percentage of P_o.
$f_i =$ Iron space factor.
$f_c =$ Copper space factor.
$d_i =$ Flux density in lines $\times 10^{-4}$ per cm.^{2} $\left(\frac{B}{10000}\right)$ (average value).
$d_c =$ Current density in ampere lines per mm.^{2} (average value).
$w_i =$ Watts per kg. iron loss (average value).
$w_c =$ Watts per kg. copper loss (average value).
$N =$ Frequency in cycles per second.
$l_i =$ Price of iron in £ per kg.
$l_c =$ Price of copper in £ per kg.
$x =$ Window width ...
$y =$ Core width ... ...
$z =$ Window height ...
$u =$ Depth of laminations

(For a core transformer, the dimensions being in decimetres.)

If we make the substitution $\begin{pmatrix} y & x & u & z & 1 \div m'' & m' \\ x & y & z & u & m' & 1 \div m'' \end{pmatrix}$ throughout, the results obtained apply where—

$$\left.\begin{array}{l} x = \text{Core width} \quad \ldots \quad \ldots \\ y = \text{Window width} \quad \ldots \\ z = \text{Depth of laminations} \\ u = \text{Window height} \quad \ldots \end{array}\right\} \text{For a shell transformer.}$$

Further—

$$h = 11 \, . \, f_i \, . \, f_c \, . \, d_i \, . \, d_c \frac{N}{50}.$$

$$g_i = 2 \, . \, 7, 7 \, . \, f_i \, . \, w_i.$$

$$g_c = 2 \, . \, 8, 9 \, . \, f_c \, . \, w_c.$$

$$k_i = 2 \, . \, 7, 7 \, . \, f_i \, . \, l_i.$$

$$k_c = 2 \, . \, 8, 9 \, . \, f_c \, . \, l_c.$$

$$4 \, F^4 = \frac{\pi}{4} c^4 = \frac{P_o}{h}.$$

$$a = \frac{g_i}{10 \, . \, h \, . \, p_i}.$$

$$b = \frac{g_c}{10 \, . \, h \, . \, p_c}.$$

$$m' \cong \frac{1}{2} \pi.$$

$$m'' \cong \frac{4}{\pi}.$$

$$m = \frac{m' + m''}{2}.$$

$$M' = \frac{b}{a} m'.$$

$$M'' = \frac{b}{a} m''.$$

$$M = \frac{M' + M''}{2}.$$

$$L^2 = \frac{a}{b} \; \frac{c^4}{a^4}.$$

The transformation is employed—

$$x = a \, X \qquad z = a \, Z$$
$$y = b \, Y \qquad u = b \, U.$$

SECTION 2.

The following expressions may be written down :—

Effective iron cross-section ... $= f_i \, . \, y \, . \, u$ dm.^{2}.
Mean iron circuit ... ... $= 2 \, (x + z + m' y)$ dm.
Volume of iron ... ... $= 2 \, . \, f_i \, . \, y \, . \, u \, (x + z + m' y)$ dm.^{3}.
Weight of iron... ... ... $= 2 \, . \, 7, 7 \, . \, f_i \, . \, y \, . \, u \, . \, (x + z + m' y)$ kgms.
Total iron loss... ... ... $= (2 \, . \, 7, 7 \, . \, f_i \, . \, w_i) \, . \, y \, . \, u \, . \, (x + z + m' y)$ wts.
$= g_i \, . \, y \, . \, u \, . \, (x + z + m' y)$ watts.

Effective copper cross-section $= f_c \,.\, x \,.\, z$ dm.^{2}.

Mean copper circuit ... ... $= 2\left(y + u + \frac{x}{m''}\right)$ dm.

Volume of copper ... ... $= 2 \,.\, f_c \,.\, x \,.\, z \,.\, \left(y + u + \frac{x}{m''}\right)$ dm.^{3}.

Weight of copper ... ... $= 2 \,.\, 8{,}9 \,.\, f_c \,.\, x \,.\, z \,.\, \left(y + u + \frac{x}{m''}\right)$ kgms.

Total copper loss ... ... $= (2 \,.\, 8{,}9 \,.\, f_c \,.\, w_c)\, x \,.\, z \,.\, \left(y + u + \frac{x}{m''}\right)$ wts.

$$= g_c \,.\, x \,.\, z \,.\, \left(y + u + \frac{x}{m''}\right) \text{ watts.}$$

The iron and copper losses are thus expressed in terms of the weight in kilograms multiplied by the watts per kilogram.

The losses may also be expressed as percentages of P_o.

$$\text{Iron loss} = \frac{p_i}{100} \,.\, 1000\, P_o \text{ watts,}$$
$$= 10\, p_i\, P_o.$$

Equating this to the former expression—

$$10\, p_i\, P_o = g_i \,.\, y \,.\, u \,.\, (x + z + m' y) \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

and in exactly the same way—

$$10\, p_c\, P_o = g_c \,.\, x \,.\, z \,.\, (y + u + \frac{x}{m''}) \quad . \; . \; . \; . \; . \; . \; . \quad (2)$$

The total flux carried by the core is—

$$f_i \,.\, y \,.\, u \,.\, d_i \,.\, 10^6.$$

The total current carried by the coils is—

$$f_c \,.\, x \,.\, z \,.\, d_c \,.\, 10^4.$$

Since the total power—that is, the sum of the input and output—is the total current multiplied by the total flux with which it is interlinked, an expression for P_o may be written down—

$$P_o = \frac{1}{2} \cdot 4{,}\, 4 \cdot 50 \cdot \frac{N}{50} \cdot f_i \cdot f_c \cdot d_i \cdot d_c \,.\, x \,.\, y \,.\, z \,.\, u \,.\, 10^6 \,.\, 10^4 \,.\, 10^{-8} \,.\, 10^{-3},$$

$$= 11 \,.\, \frac{N}{50} \cdot f_i \cdot f_c \,.\, d_i \,.\, d_c \,.\, x \,.\, y \,.\, z \,.\, u \text{ kilovolt-amperes} \quad . \; . \; . \quad (3)$$

SECTION 3.

Using the values of a, b, and c^4 (1) becomes—

$$a \,.\, y \,.\, u \,.\, (x + z + m' y) = x \,.\, y \,.\, z \,.\, u \quad . \; . \; . \; . \; . \; . \quad (4)$$

(2) becomes—

$$b \,.\, x \,.\, z \,.\, (y + u + \frac{x}{m''}) = x \,.\, y \,.\, z \,.\, u \quad . \; . \; . \; . \; . \; . \quad (5)$$

(3) becomes—

$$x \cdot y \cdot z \cdot u = \frac{c^4}{m''} \quad \dots \dots \quad (6)$$

After some reduction (4), (5) may be written in the form—

$$z = \frac{a\,(x + m'\,y)}{x - a} \quad \dots \dots \quad (7)$$

$$u = \frac{b\,(x + m''\,y)}{m''\,(y - b)} \quad \dots \dots \quad (8)$$

Multiplying (7) by (8) and using (6)—

$$z\,u = \frac{c^4}{m''\,x\,y} = \frac{a\,b\,(x + m'\,y)\,(x + m''\,y)}{m''\,(x - a)\,(y - b)}.$$

That is—

$$x\,y\,(x + m'\,y)\,(x + m''\,y) = \frac{c^4}{a\,b}\,(x - a)\,(y - b) \quad \dots \quad (9)$$

SECTION 4.

The system of equations—

$$\begin{cases} z = \dfrac{a\,(x + m'\,y)}{x - a} & \dots \dots \quad (7) \\ u = \dfrac{b\,(x + m''\,y)}{m''\,(y - b)} & \dots \dots \quad (8) \\ x\,y\,(x + m'\,y)\,(x + m''\,y) = \dfrac{c^4}{a\,b}\,(x - a)\,(y - b) & \dots \quad (9) \end{cases}$$

gives the fundamental relations between the dimensions of a transformer when all the other elements of design are kept constant.

This assumption may be justified by supposing that a fairly close design has been already obtained by slide-rule work, and that for the range between this design and the best dimensions the variation in densities and space factors is negligibly small.

It is seen from the system that there are three relations between four variables, and that there is therefore a onefold infinity of values of $x\,y\,z\,u$, satisfying the system.

In fact, any values of x and y satisfying (9) when substituted in (7) and (8) give values of z and u, and the values of $x\,y\,z$ and u so obtained give the dimensions of a transformer satisfying the data of design.

SECTION 5.

The general form of the curve is shown in Fig. 1.*

Now it is clear that $x\,y\,z$ and u must be real and positive, hence from equations (5), (6) we must have—

$$x > a$$
$$y > b.$$

---

* NOTE.—Cf. with some interesting quartics with and without oval branches discussed in Frost's "Curve-Tracing."

It is seen from Fig. I that the oval branch lying beyond the straight lines $x = a$, $y = b$ is the only branch leading to positive values of $x\,y\,z$ and u.

It is therefore necessary only to consider the variation of the oval branch.

SECTION 6.

Let the ratio of p_i to p_c be kept constant.

It follows that $p_i \propto p_c \propto \frac{1}{a} \propto \frac{1}{b}$.

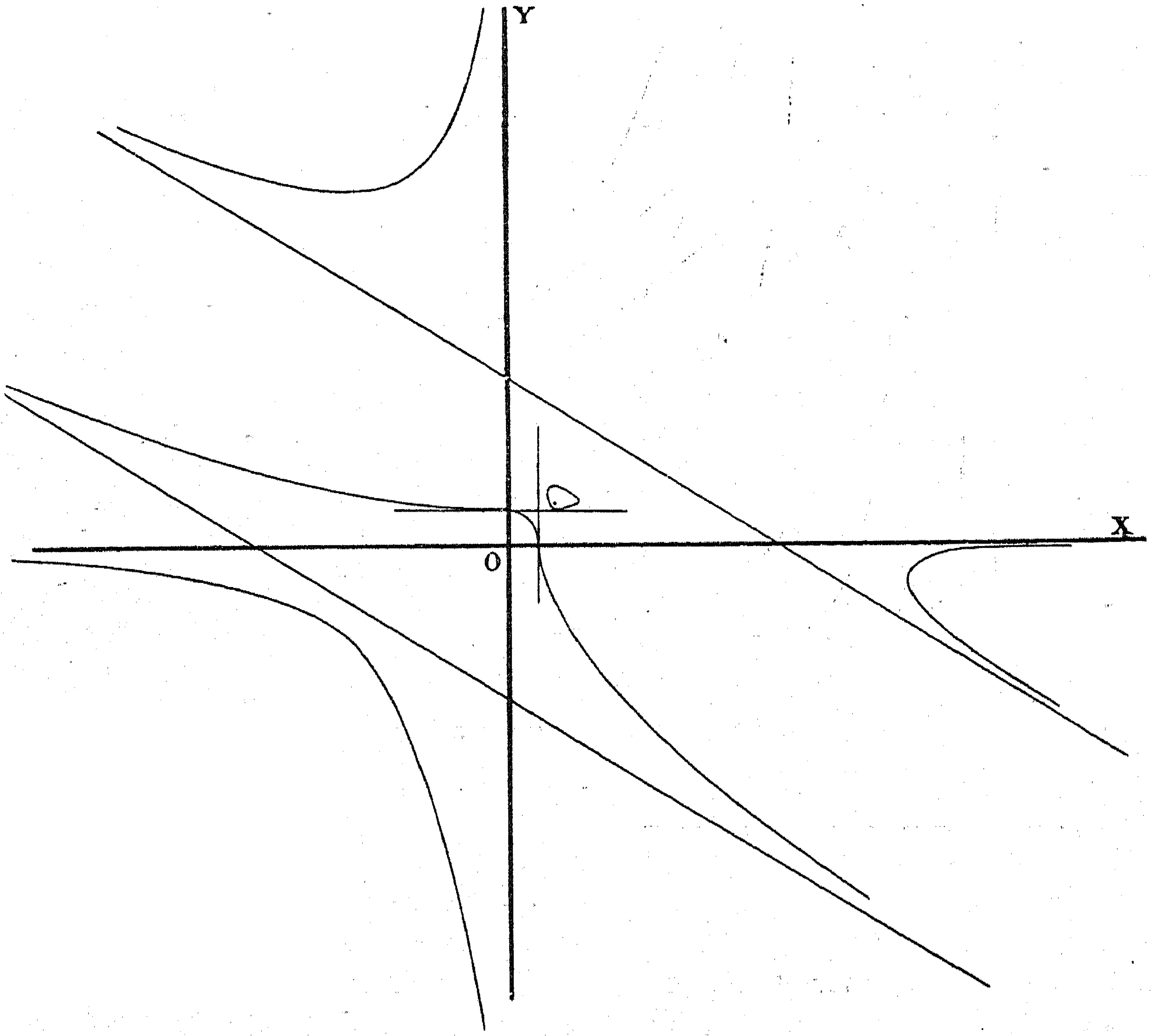

FIG. I.

It is found that as p_i increases the oval swells until it coincides with the straight lines $x = o$, $y = o$ and the straight line at infinity, and as p_i decreases the oval shrinks until for definite values of p_i, p_c and the corresponding values of a and b it becomes a point.

No oval cuts any other oval, hence from geometrical considerations the point is a conjugate point.

This point gives unique values of x and y, from which unique values of z and u are derived by equations (7) and (8) taking the corresponding values of a and b.

For smaller values of p_i, p_c the oval branch disappears.

The values of $x\ y$ at the conjugate point and the derived values of $z\ u$ constitute a unique solution of the transformer dimensions for the given ratio $p_i : p_c$ and for minimum values of p_i and p_c.

SECTION 7.

Let now the ratio p_i to p_c vary continuously, the conjugate point also moves continuously, tracing out a locus of conjugate points.

Finally it rises to be determined what point on this locus corresponds to the transformer of least cost and of least losses, independently of the ratio of p_i to p_c.

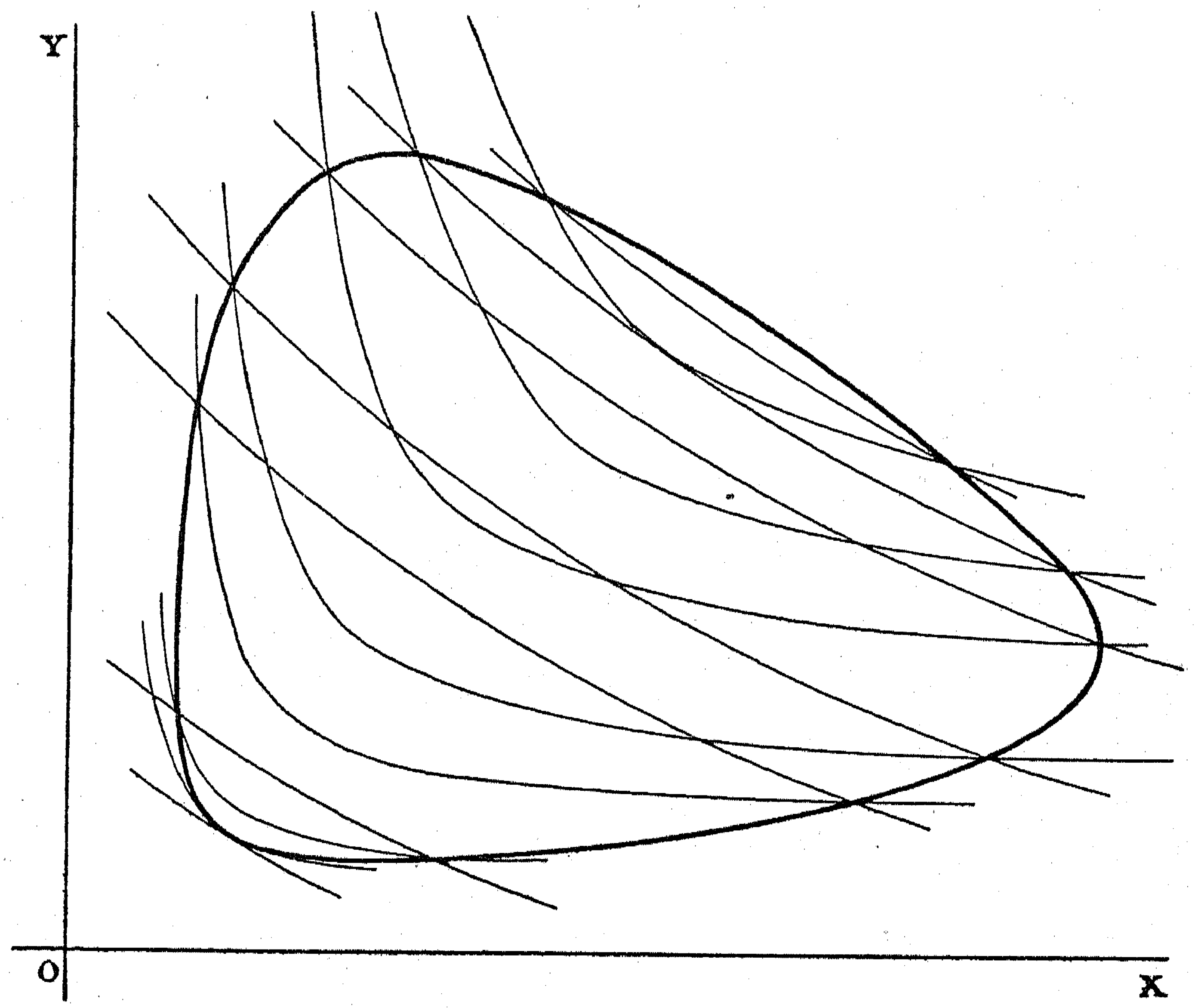

FIG. 2.

SECTION 8.

The analysis is much simplified by the transformation to X Y Z U.

(7) becomes—

$$Z = \frac{X + M'Y}{X - 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

(8) becomes—

$$U = \frac{X + M''Y}{M''(Y - 1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

(9) becomes—

$$XY(X + M'Y)(X + M''Y) = L^2 (X - 1 (Y - 1) \quad . \quad (12)$$

The quartic (12) may be conveniently plotted as the intersection of the two families of hyperbolas—

$$(X + M'Y)(X + M''Y) = L^2 \sin^2 i$$

and—

$$XY \sin^2 i = (X - 1)(Y - 1),$$

which reduces to—

$$(X - \sec^2 i)(Y - \sec^2 i) = \sec^2 i \tan^2 i,$$

where i is a variable parameter (see Fig. 2).

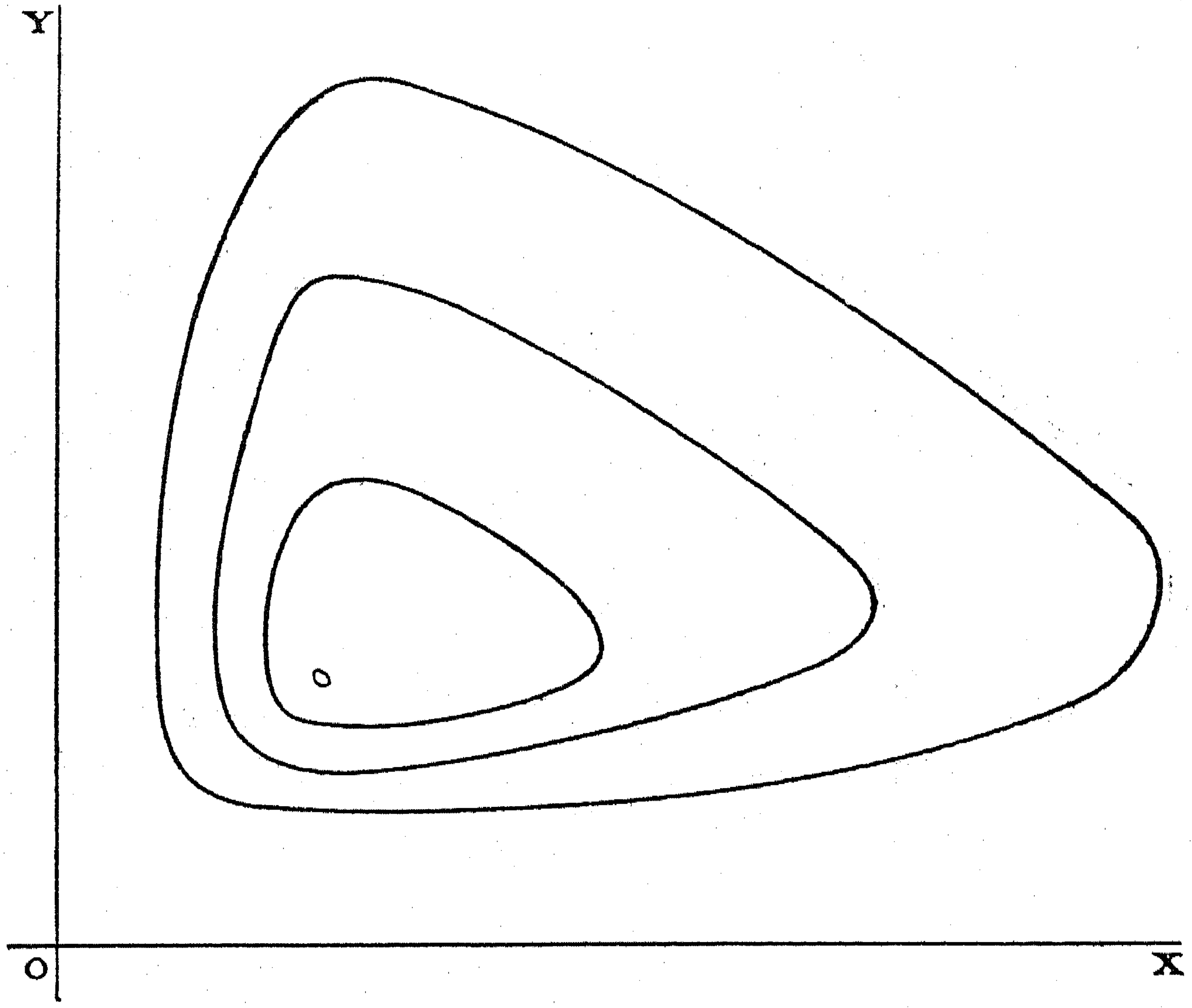

FIG. 3.

Different values of a give different values of $L^2 \left(\frac{a}{b} = \text{const.}\right)$ corresponding to different ovals (see Fig. 3).

SECTION 9.

To find the locus of conjugate points we may rewrite (12), dividing by XY—

$$V \equiv (X + M'Y)(X + M''Y) - L^2\left(1 - \frac{1}{X}\right)\left(1 - \frac{1}{Y}\right) = 0 \quad . \quad . \quad (13)$$

and the conditions for a singular point—

$$\frac{dV}{dx} \equiv (X + M'Y) + (X + M''Y) - L^2\frac{1}{X^2}\left(1 - \frac{1}{Y}\right) = 0 \quad . \quad . \quad . \quad (14)$$

$$\frac{dV}{dy} \equiv M\,(X + M''Y) + M''(X + M'Y) - L^2\frac{1}{Y^2}\left(1 - \frac{1}{X}\right) = 0 \quad . \quad (15)$$

Add (14) multiplied by X to (15) multiplied by Y—

$$2\,(X + M'Y)\,(X + M''Y) - L^2\left\{\frac{1}{X}\left(1 - \frac{1}{Y}\right) + \frac{1}{Y}\left(1 - \frac{1}{X}\right)\right\} = 0.$$

Using (13) this reduces to—

$$2\left(1 - \frac{1}{X}\right)\left(1 - \frac{1}{Y}\right) - \frac{1}{X}\left(1 - \frac{1}{Y}\right) - \frac{1}{Y}\left(1 - \frac{1}{X}\right) = 0.$$

After further reduction—

$$\left(X - \frac{3}{2}\right)\left(Y - \frac{3}{2}\right) = \frac{1}{4} \text{ (see Fig. 4)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Since the equation is free of L^2, M', and M'' it is the locus of singular points.

Hence, noting that the lower branch passes through the point R (1, 1) at an angle of 45° with the negative directions of the axes, it is seen that only the upper branch satisfies the conditions of Section 5—

$$x > a \qquad y > b.$$

That is—

$$X > 1 \qquad Y > 1.$$

It is clear from physical and geometrical considerations that the singular points on the upper branch are conjugate points.

Dr. L. N. G. Filon has furnished an analytical proof, which is not printed here.

SECTION 10.

If to (14) multiplied by M′ + M″ we add (15) multiplied by —2

$$(M' - M'')\,(X + M'Y - X - M''Y) = L^2\left\{\frac{1}{X^2}\left(1 - \frac{1}{Y}\right)(M' + M'') - \frac{2}{Y^2}\left(1 - \frac{1}{X}\right)\right\}$$

after some reduction—

$$\left(X - \frac{1}{2}\right)^2 - \frac{M' + M''}{2}\left(Y - \frac{1}{2}\right)^2 = \frac{1}{4}\left(1 - \frac{M' + M''}{2}\right) + (M' - M'')^2\frac{X^2Y^3}{2L^2} \quad . \quad . \quad (17)$$

The intersection of (17) and (16) gives the values of X Y corresponding to the given values of M′ M″.

SECTION 11.

Graphical solution of the particular case where $M' = M''$.

Equation (17) now becomes—

$$\left(X - \frac{1}{2}\right)^2 - M\left(Y - \frac{1}{2}\right)^2 = \frac{1}{4}(1 - M) \quad . \quad . \quad . \quad (18)$$

Curves (16) and (18) are shown in Fig. 4 intersecting in four points P, Q, R, S, of which only P lies on the upper branch of (16).

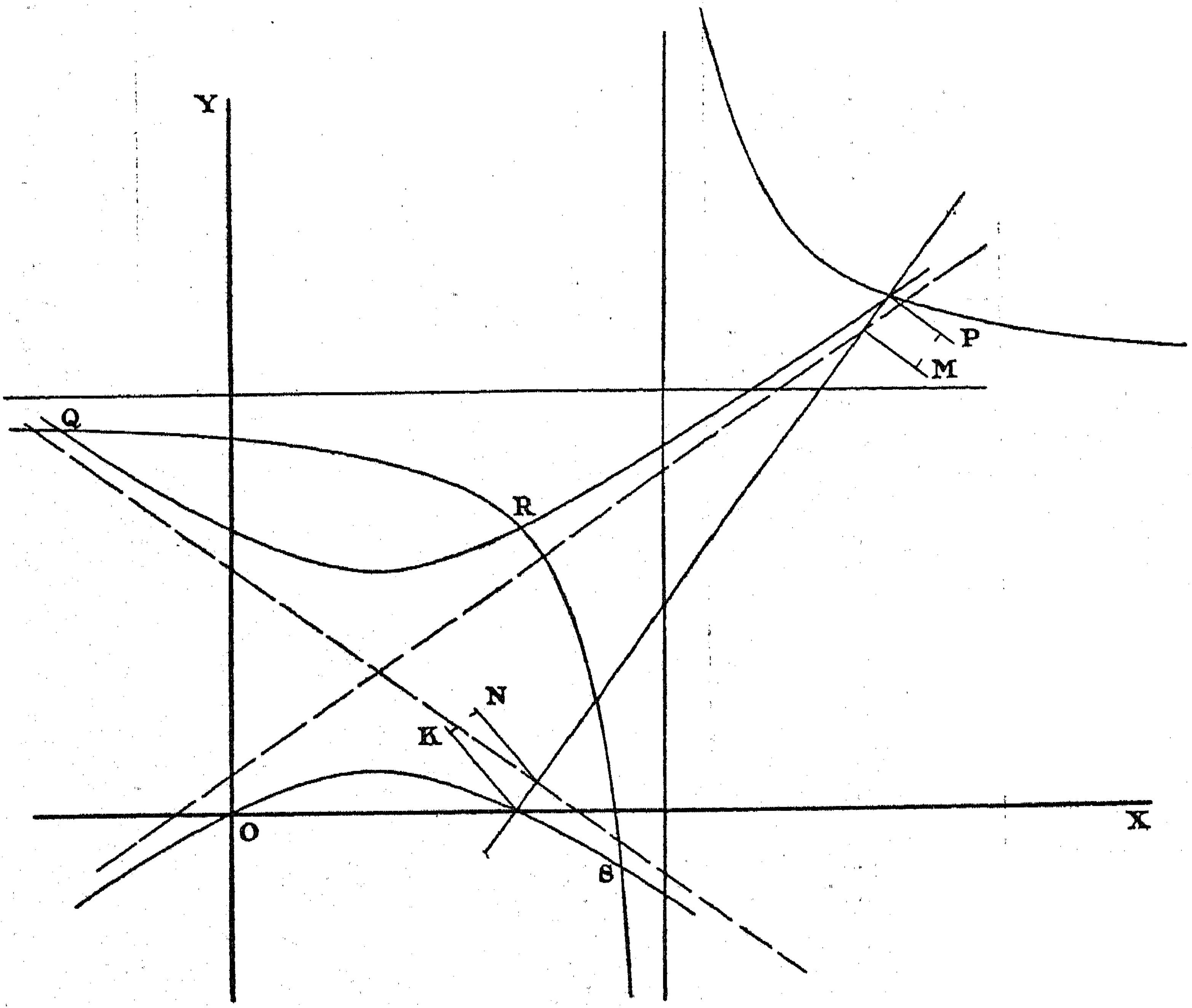

FIG. 4.

To find P without drawing (18), draw the asymptotes—

$$\left(X - \frac{1}{2}\right) = \pm \sqrt{M}\left(Y - \frac{1}{2}\right)$$

and lay a straight edge across the figure so as to make $PM = KN$, K being the point (1, 0).

SECTION 12.

Taking X as the independent variable, Y and M may be calculated and tabulated.

Rearranging the columns, we get values of X and Y tabulated against M.

M.	X.	Y.
12·50	4·00	1·600
12·08	3·95	1·600
11·66	3·90	1·605
11·25	3·85	1·605
10·85	3·80	1·610
10·46	3·75	1·610
10·07	3·70	1·615
9·70	3·65	1·615
9·34	3·60	1·620
8·98	3·55	1·620
8·61	3·50	1·625
8·26	3·45	1·630
7·92	3·40	1·630
7·58	3·35	1·635
7·25	3·30	1·640
6·93	3·25	1·645
6·61	3·20	1·645
6·30	3·15	1·650
6·00	3·10	1·655
5·70	3·05	1·660
5·41	3·00	1·665
5·13	2·95	1·675
4·85	2·90	1·680
4·58	2·85	1·685
4·32	2·80	1·690
4·06	2·75	1·700
3·81	2·70	1·710
3·57	2·65	1·715
3·33	2·60	1·725
3·10	2·55	1·740
2·87	2·50	1·750
2·65	2·45	1·765
2·44	2·40	1·780
2·24	2·35	1·795
2·04	2·30	1·810
1·85	2·25	1·830
1·67	2·20	1·855
1·49	2·15	1·885
1·32	2·10	1·915
1·16	2·05	1·955
1·00	2·00	2·000
$\frac{1}{M}$.	Y.	X.

L^2 might also be tabulated, but it is calculable directly from X, Y, and M by means of equations (13), (14), (15), each of which should give the same value of L^2 if the graphical work or tabulation is accurate.

SECTION 13.

From (13) and (14) by division—

$$\frac{(X + M Y')(X + M'' Y)}{X + M' Y + X + M'' Y} = X^2 \left(1 - \frac{1}{X}\right).$$

∴ From (10)—

$$Z = \frac{X + M' Y}{X - 1} = \frac{X\,[2\,X + (M' + M'')\,Y]}{X + M''\,Y}.$$

$$Z = 2\,X\left[1 + \frac{(M' - M'')\,Y}{2(X + M''\,Y)}\right].$$

$$z = 2\,x\left[1 + \frac{(m' - m'')\,y}{2\,(x + m''\,y)}\right] \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (19)$$

From (13) and (15) by division—

$$\frac{(X + M' Y)(X + M'' Y)}{M''(X + M' Y) + M'(X + M'' Y)} = Y^2 \left(1 - \frac{1}{Y}\right).$$

∴ From (9)—

$$U = \frac{X + M'' Y}{M''\,(Y - 1)} = \frac{Y\,[M'\,(X + M''\,Y) + M''\,(X + M'\,Y)]}{M''\,(X + M'\,Y)}$$

$$U = 2\,Y\left[1 + \frac{(M' - M'')\,X}{2\,M''\,(X + M'\,Y)}\right]$$

$$u = 2\,y\left[1 + \frac{(m' - m'')\,x}{2\,m''(x + m'\,y)}\right] \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (20)$$

SECTION 14.

When $M' = M''$ (19) becomes—

$$Z = 2\,X$$

$$z = 2\,x \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (21)$$

(20) becomes—

$$U = 2\,Y$$

$$u = 2\,y \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (22)$$

Hence we have a complete solution in the particular case where $M' = M''$ of the dimensions corresponding to any given ratio of the losses.

Since $M' - M''$ is usually small, the solution obtained by putting $M = \frac{M' + M''}{2}$ for both M' and M'' throughout will give an approximate solution of the general case, equations (17), (19), (20) enabling us to examine the degree of approximation.

SECTION 15.

A complete algebraic solution is also found, in the particular case, for the problems of Section 7.

Taking the problem of finding the most efficient transformer, and putting $m' = m'' = m$, we may write down the total losses—

$$Q \equiv g_i . y . u . (x + z + m y) + g_c . x . z . \left(y + u + \frac{x}{m}\right) \quad . \quad . \quad (23)$$

From (6)—

$$4 R^2 \equiv x . y . z . u - 4 F^4 = 0,$$

using (21), (22).

$$Q \equiv 2 g_i y^2 (3 x + m y) + 2 g_c x^2 \left(3 y + \frac{x}{m}\right) \quad . \quad . \quad . \quad (24)$$

$$R^2 \equiv x^2 y^2 - F^4 = 0.$$

$$R \equiv x y - F^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The condition that Q is a minimum subject to $R = 0$ is—

$$\frac{d Q}{d x} \frac{d R}{d y} - \frac{d Q}{d y} \frac{d R}{d x} = 0,$$

or—

$$x \left(6 g_i y^2 + 12 g_c x y + 6 g_c \frac{x^2}{m}\right) - y (6 g_i m y^2 + 12 g_i x y + 6 g_c x^2) = 0.$$

After some reduction—

$$6 (x + m y) \left(g_c \frac{x^2}{m} - g_i y^2\right) = 0,$$

rejecting the solution $x + m y = 0$—

$$\frac{x^2}{y^2} = \frac{m g_i}{g_c}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

whence by (24)—

$$Q \equiv 2 g_i y^2 \left\{(3 x + m y) + (3 m y + x)\right\},$$

and the ratio of iron and copper losses is—

$$\frac{3 x + m y}{3 m y + x}$$

a quantity whose upper and lower limits are 3 and $\frac{1}{3}$.

By (25) and (26)—

$$\frac{x^4}{F^4} = \frac{m g_i}{g_c},$$

rejecting all negative roots—

$$\left.\begin{aligned} x &= \sqrt[4]{\frac{m g_i}{g_c}}\, F \\ y &= \sqrt[4]{\frac{g_c}{m g_i}}\, F \\ z &= 2 x \\ y &= 2 w \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

SECTION 16.

If k_i, k_c be substituted for g_i, g_c in (27), we get the dimensions of the cheapest transformer as far as cost of active materials goes.

It is seen at once that the two solutions are the same only if $\frac{k_i}{k_c} = \frac{g_i}{g_c}$, and not otherwise.

This solution is evidently an approximate solution of the general case when $M' - M''$ is small.

SECTION 17.

It has hitherto been supposed that both copper and iron densities are rated up as high as possible, leaving a reasonable margin of safety.

Under these circumstances the assumption is justified that the flux and current densities are constant for a pretty wide range of dimensions.

In British practice another case arises where a large amount of material is put into a transformer at low densities to keep down the no-load losses without sacrificing regulation.

If we replace d_i and d_c by two variables μ and ν, we may vary μ and ν subject to the condition $\mu \times \nu =$ constant, without altering the output.

Again, if—

$$w_i = \alpha \mu^n \ (n = 1{\cdot}5 \text{ to } 2),$$
$$w_c = \beta \nu^2.$$

we may increase the efficiency by increasing all the linear dimensions in the same proportion.

It has already been established that for any values of μ and ν we can find unique values of the dimensions which give minimum losses in any arbitrary ratio.

Combining these two methods, it is clear that by varying the densities and dimensions we can obtain any values for the iron and copper losses for any ratio of μ and ν.

The new problem is, what ratio of μ and ν gives the cheapest transformer while meeting the specified losses.

The solution leads to abnormally low flux densities.

The analysis is beyond the scope of the present paper.

SECTION 18.

Example.

(50 Kilovolt-amperes Transformer, 2000/200 Volts.)

$P_0 = 50{\cdot}5$.

$d_i = 1$. $w_i = 2{\cdot}2$.

$d = 1$. $w_c = 2{\cdot}2$.

$f_i = 0{\cdot}9$.

$f_c = 0{\cdot}4$.

$N = 50$.

$$
\begin{aligned}
F^4 &= \frac{50{\cdot}5}{4 \times 11 \times 0{\cdot}9 \times 0{\cdot}4 \times 1 \times 1} = 3{\cdot}19.\\
F &= 1{\cdot}337.\\
g_i &= 2 \times 7{\cdot}7 \times 0{\cdot}9 \times 2{\cdot}2 = 30{\cdot}5.\\
g_c &= 2 \times 8{\cdot}9 \times 0{\cdot}4 \times 2{\cdot}2 = 15{\cdot}7.\\
m' &= 1{\cdot}57. \qquad m'' = 1{\cdot}27. \qquad m = 1{\cdot}42.
\end{aligned}
$$

Approximate solution—

$$
\left\{
\begin{aligned}
x_1 &= \sqrt[4]{\frac{1{\cdot}42 \times 30{\cdot}5}{15{\cdot}7}} \times 1{\cdot}337 = 1{\cdot}72.\\
z_1 &= 3{\cdot}44.\\
y_1 &= \sqrt[4]{\frac{15{\cdot}7}{1{\cdot}42 \times 30{\cdot}5}} \times 1{\cdot}337 = 1{\cdot}035.\\
u_1 &= 2{\cdot}07.
\end{aligned}
\right.
$$

The losses calculated on the supposition that $m' = m''$ are by (26)—

$$2\, g_i y_1^2 (3\, x_1 + m\, y_1) \quad \text{and} \quad 2\, g_i y_1^2 (3\, m\, y_1 + x_1).$$

$$
\begin{aligned}
\text{Iron loss} &= 2 \times 30{\cdot}5 \times 1{\cdot}035^2 \times 6{\cdot}63 = 430 \text{ watts.}\\
\text{Copper loss} &= 2 \times 30{\cdot}5 \times 1{\cdot}035^2 \times 6{\cdot}12 = 400 \text{ watts.}
\end{aligned}
$$

The actual losses for the above values of $x\, y\, z\, u$ are—

$$
\begin{aligned}
\text{Iron loss} &= 2\, g_i y_1^2 (3\, x_1 + m' y_1) = 440 \text{ watts.}\\
\text{Copper loss} &= 2\, g_c x_1^2 \left(3\, y_1 + \frac{x_1}{m''}\right) = 415 \text{ watts.}
\end{aligned}
$$

The ratio of actual losses is $\dfrac{3\, x_1 + m' y_1}{3\, m\, y_1 + \dfrac{m}{m''}\, x_1}$, which here $= 0{\cdot}95$, and, in general, is not necessarily unity.

We may test the degree of approximation by means of (19) and (20). A second approximation is—

$$
\begin{aligned}
z &= 2\, x \left[1 + \frac{0{\cdot}3 \times 1{\cdot}04}{2\,(3{\cdot}03)}\right].\\
&= 2\, x\, [1 + 0{\cdot}052].\\
u &= 2\, y \left[1 + \frac{0{\cdot}3 \times 1{\cdot}72}{2 \times 1{\cdot}27 \times 3{\cdot}35}\right].\\
&= 2\, y\, [1 + 0{\cdot}061].
\end{aligned}
$$

Hence the dimensions are within 5 or 6 per cent. of the theoretically best design, and as we are in the neighbourhood of a turning value the total losses will not be out by more than a small quantity of the order of $(0{\cdot}05)^2$ or $(0{\cdot}06)^2$—that is, ¼ or ⅓ per cent.

SECTION 19.

Example of Graphical Method and Use of the Table.

$$P_o = 40{\cdot}5.$$
$$N = 50.$$
$$p_i : p_c = 1 : 2{\cdot}5$$
$$f_i = 0{\cdot}9.$$
$$f_c = 0{\cdot}35.$$
$$d_i = 1. \qquad w_i = 2. \qquad g_i = 27{\cdot}6.$$
$$d_c = 1. \qquad w_c = 2{\cdot}22. \qquad g_c = 13{\cdot}8.$$
$$c^4 = \frac{4}{\pi} \times \frac{40{\cdot}5}{11 \times 0{\cdot}9 \times 0{\cdot}35 \times 1 \times 1} \times \frac{50}{50} = 15.$$
$$a = \frac{2 \times 7{\cdot}7 \times 2}{110 \times 0{\cdot}35 \times p_i} = \frac{0{\cdot}8}{p_i}.$$
$$b = \frac{2 \times 8{\cdot}9 \times 2{\cdot}22}{110 \times 0{\cdot}9 \times p_c} = \frac{0{\cdot}4}{p_c}.$$
$$\frac{a}{b} = \frac{0{\cdot}8}{0{\cdot}4} \times \frac{p_c}{p_i} = 5.$$
$$m' = 1{\cdot}57. \qquad m'' = 1{\cdot}27. \qquad m = 1{\cdot}42.$$
$$M = m \div \frac{a}{b} = 0{\cdot}284.$$
$$\sqrt{M} = 0{\cdot}533.$$

Approximate Solution.

Drawing the line pair—

$$\left(X - \frac{1}{2}\right) = \pm \sqrt{M} \left(Y \mp \frac{1}{2}\right),$$

and laying off P M = K N as in Fig. 4, we get—

$$X = 1{\cdot}7, \qquad Y = 2{\cdot}7.$$

Or, using the table and reading from the bottom, by proportional parts—

$$\frac{1}{M} = 3{\cdot}52, \qquad Y = 2{\cdot}65, \qquad X = 1{\cdot}71.$$

Substituting these values of X Y and M in (13), (14), or (15) we get—

$$E^2 = 23{\cdot}5 = \frac{a}{b} \frac{c^4}{a^4}.$$

Whence in succession—

$$a^4 = \frac{5 \times 15}{23{\cdot}5} = 3{\cdot}2.$$
$$a = 1{\cdot}34.$$
$$b = \frac{a}{5} = 0{\cdot}268.$$
$$p_i = \frac{0{\cdot}8}{a} = 0{\cdot}6 \text{ per cent.}$$
$$p_c = \frac{0{\cdot}4}{b} = 1{\cdot}5 \text{ per cent.}$$

Iron loss = 242 watts.
Copper loss = 610 watts.

Again—

$$\begin{aligned} x &= aX = 1{\cdot}34 \times 1{\cdot}7 = 2{\cdot}28. \\ y &= bY = 0{\cdot}268 \times 2{\cdot}7 = 0{\cdot}725. \\ z &= 2x = 4{\cdot}56. \\ u &= 2y = 1{\cdot}45. \end{aligned}$$

Substituting these values in the right-hand side of (1) and (2)—

$$\begin{aligned} \text{Iron loss} &= 2 \times 27{\cdot}6 \times 0{\cdot}725^2 \times 7{\cdot}98 = 230. \\ \text{Copper loss} &= 2 \times 13{\cdot}8 \times 2{\cdot}28^2 \times 3{\cdot}97 = 570. \end{aligned}$$

There is a discrepancy of 5 or 6 per cent. between the two evaluations of the losses.

The ratio required, however, is accurately enough attained.

SECTION 20.

I am specially indebted to Dr. L. N. G. Filon, to whom I owe the method of Sections 9 and 10, and the elegant graphical solution of paragraph 11; and my thanks are due to Professor Kapp, without whose encouragement the work would at one time have been laid aside.

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